Boundary controllability of some coupled parabolic systems with Kirchhoff conditions

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a joint work with

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Outline

Outline

Parabolic control problems on a graph

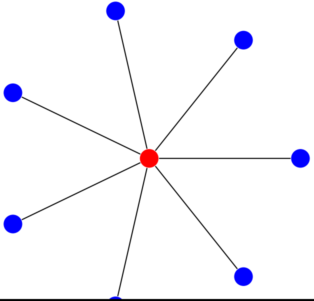
Consider the following system of n PDEs

$$(1) \qquad \begin{cases} \partial_t y + \mathcal{A} y = 0 & \text{in } (0,T)\times(0,1), \\ y(0,x) = y_0(x) & \text{in } (0,1), \end{cases}$$

where $y = (y_1, y_2, \cdots, y_n)$ and the boundary conditions

$$(2) \qquad \begin{cases} y_j(t,0) = v_j(t), \quad j = 0,1,\cdots,(n-1), \quad y_n(t,0) = 0, & \text{for } t \in (0,T), \\ y_1(t,1) = y_2(t,1) = \cdots = y_n(t,1), & \text{for } t \in (0,T), \\ \displaystyle\sum_{j=1}^{n} \partial_x y_j(t,1) = 0, & \text{for } t \in (0,T). \end{cases}$$

Null-controllability. The above system is null-controllable for any initial data $y_0 \in (H^{-1}(0,1))^n$ and given time $T > 0$.



Red node: *Kirchhoff conditions*,

Blue nodes: *Dirichlet conditions*.

This kind of control systems has already been appeared for instance, in the book by Dáger, Zuazua, 2006; the survey paper by Avdonin, 2008.

Outline

Problem statement and main results

[Bhandari, Boyer, Hernández-Santamaría (submitted), 2020]

Consider the following 1-D system

$$
(3) \quad \begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a\, y_1 = 0 & \text{in } (0,T)\times(0,1), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha\, y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases}
$$

$a \in \mathbb{R}, \alpha \geq 0$, with the following two situations

$$
(4a) \quad \text{either} \quad y_1(t,0) = v(t), \quad y_2(t,0) = 0 \quad \text{in } (0,T),
$$

$$
(4b) \quad \text{or} \quad y_1(t,0) = 0, \quad y_2(t,0) = v(t) \quad \text{in } (0,T).
$$

- The diffusion coefficients $\gamma_i \in \mathcal{C}^1([0,1])$ with

$$
\gamma_{\min} := \inf_{[0,1]} \{\gamma_i;\ i = 1,2\} > 0.
$$

Functional setting

- Denote $\mathcal{A}_\alpha = \begin{pmatrix} -\partial_x(\gamma_1\partial_x) & 0 \\ 0 & -\partial_x(\gamma_2\partial_x) \end{pmatrix}$, with its domain

$$D(\mathcal{A}_\alpha) := \Big\{ u = (u_1, u_2) \in (H^2(0,1))^2 \mid u_1(0) = u_2(0) = 0,$$
$$u_1(1) = u_2(1), \quad \gamma_1(1)u_1'(1) + \gamma_2(1)u_2'(1) + \alpha u_1(1) = 0 \Big\}.$$

The coupling matrix $\mathcal{M}_a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ and the operator

$$\mathcal{A}_{\alpha,a} = \mathcal{A}_\alpha + \mathcal{M}_a = \begin{pmatrix} -\partial_x(\gamma_1\partial_x) & 0 \\ a & -\partial_x(\gamma_2\partial_x) \end{pmatrix},$$

with its domain $D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$.

- We denote the adjoint operator $\mathcal{A}^*_{\alpha,a}$ with $D(\mathcal{A}^*_{\alpha,a}) = D(\mathcal{A}_{\alpha,a})$.
- Further, we consider the space $\mathcal{H} := D(\mathcal{A}_\alpha^{1/2})$ as a completion of $D(\mathcal{A}_\alpha)$ w.r.t. the norm

$$(5) \qquad \|u\|_{\mathcal{H}} = (\mathcal{A}_\alpha u, u)_{L^2}^{\frac{1}{2}} = \left(\sum_{i=1}^{2} \int_0^1 \gamma_i(x)|u_i'(x)|^2 dx + \alpha |u_1(1)|^2 \right)^{\frac{1}{2}}, \ \forall u \in D(\mathcal{A}_\alpha).$$

Indeed,

$$\mathcal{H} = \Big\{ u = (u_1, u_2) \in (H^1(0,1))^2 \mid u_1(0) = u_2(0) = 0, \ u_1(1) = u_2(1) \Big\}.$$

- Denote the dual space of $\mathcal{H}$ by $\mathcal{H}_{-1}$ w.r.t. the pivot space $(L^2(0,1))^2$.
- The observation operators are (for 1st & 2nd cases)

$$\mathcal{B}_i^* : u = (u_1, u_2) \in (H^2(0,1))^2 \mapsto \gamma_i(0)u_i'(0), \quad \text{(resp. for } i = 1, 2).$$

Main Theorems

Theorem (Control on y_2)

Let any $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}$ and $T > 0$ be given. Then, for any $y_0 \in \mathcal{H}_{-1}$, there exists a null-control $v \in L^2(0,T)$ for the problem (3)–(4b), that satisfies the estimate

(6)
$$\|v\|_{L^2(0,T)} \leq Ce^{C/T}\|y_0\|_{\mathcal{H}_{-1}},$$

with the constant $C := C(\gamma_1, \gamma_2, \alpha, a) > 0$ that does not depend on $T > 0$ and y_0.

Theorem (Control on y_1)

We fix $\gamma_1 = \gamma_2 = 1$. Then, there exists a set $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$ such that

1. *for each pair $(\alpha, a) \notin \mathcal{R}$, there is a null-control to the problem (3)–(4a), for any given data $y_0 \in \mathcal{H}_{-1}$,*
2. *for each pair $(\alpha, a) \in \mathcal{R}$, there exists a subspace $\mathcal{Y}_{\alpha,a} \subset \mathcal{H}_{-1}$ of co-dimension 1, such that there exists a null-control to the problem (3)–(4a), if and only if $y_0 \in \mathcal{Y}_{\alpha,a}$.*

In addition, in the controllable cases we can choose such a null-control $v \in L^2(0,T)$ that satisfies the bound

(7)
$$\|v\|_{L^2(0,T)} \leq C_{\alpha,a}\, e^{C_{\alpha,a}/T}\|y_0\|_{\mathcal{H}_{-1}},$$

where $C_{\alpha,a} > 0$ is independent on $T > 0$ and y_0.

$\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.

Outline

Control on y_2: a Carleman approach

The system with *control on* y_2

$$(8)\quad \begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a\, y_1 = 0 & \text{in } (0,T)\times(0,1), \\ y_1(t,0) = 0, \quad y_2(t,0) = v(t) & \text{in } (0,T), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1). \end{cases}$$

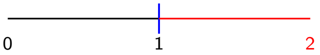
The system with $a = 0$, $\alpha = 0$ is equivalent to the scalar problem

$$(9)\quad \begin{cases} \partial_t y - \partial_x(\gamma \partial_x y) = 0 & \text{in } (0,T)\times(0,2), \\ y(t,0) = v(t), \quad y(t,2) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,2), \end{cases}$$

with $\gamma(x) = \begin{cases} \gamma_1(x) & x \in (0,1) \\ \gamma_2(2-x) & x \in (1,2) \end{cases}$ and the following conditions at $x = 1$,

$$(10)\quad \gamma_1(1)\partial_x y(t,1) = \gamma_2(1)\partial_x y(t,1).$$

This has been addressed by Benabdallah, Dermenjian, Le Rousseau, 2007.



Idea to construct Carleman weight functions

The scalar system

$$\begin{cases} \partial_t y - \partial_x(\gamma \partial_x y) = 0 & \text{in } (0,T)\times(0,2), \\ y(t,0) = v(t), \quad y(t,2) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,2). \end{cases}$$

$$\gamma(x) = \begin{cases} \gamma_1(x) & x \in (0,1) \\ \gamma_2(2-x) & x \in (1,2) \end{cases}$$

with the following conditions at $x = 1$

$$\gamma_1(1)\partial_x y(t,1) = \gamma_2(1)\partial_x y(t,1).$$

The following type of auxiliary function has been used to construct the weight functions in the paper Benabdallah, Dermenjian, Le Rousseau, 2007.

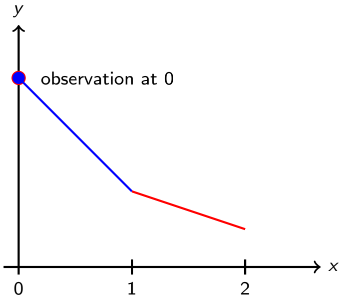


Figure: Typical auxiliary function: Case of a discont. coeff.

Carleman weight functions

- Denote $Q := (0, T) \times (0, 1)$. We rewrite the system with control on y_2

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } Q, \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a\, y_1 = 0 & \text{in } Q, \\ y_1(t,0) = 0, \quad y_2(t,0) = v(t) & \text{in } (0,T), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha\, y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases}$$

with $a \in \mathbb{R}$, $\alpha \geq 0$.

- To construct the **weight functions** we use the **affine functions** as:

$$\begin{cases} \beta_i(x) = 2 + c_i(x-1), \quad \forall x \in [0,1], \\ \text{with } \ c_1 = 1, \quad c_2 := c_2(\gamma_1, \gamma_2) < 0 \ \text{ with large enough modulus} \\ \beta_2 \geq \beta_1 > 0, \ \text{in } [0,1], \quad \beta_2(1) = \beta_1(1). \end{cases}$$

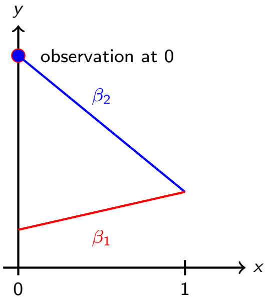


Figure: Typical auxiliary function

Carleman estimate

Assume that $\lambda > 1$ and $K = 2\|\beta_2\|_\infty$. Define the weight functions

$$\varphi_i(t,x) = \frac{e^{\lambda\beta_i(x)}}{t(T-t)}, \quad \eta_i(t,x) = \frac{e^{\lambda K} - e^{\lambda\beta_i(x)}}{t(T-t)}, \quad \forall (t,x) \in Q, \quad i = 1,2.$$

Introduce the space $\mathcal{Q} := \Big\{ q = (q_1, q_2) \in (\mathcal{C}^2(\overline{Q}))^2 \mid q_1(t,0) = q_2(t,0) = 0,\ q_1(t,1) = q_2(t,1),$

$$\sum_{i=1}^{2} \gamma_i(1)\partial_x q_i(t,1) + \alpha q_1(t,1) = 0,\ \forall t \in (0,T) \Big\}.$$

Theorem

There exists $\lambda_1 := \lambda_1(\gamma_1,\gamma_2,\alpha) > 0$, $s_1 := (T^2+T)\sigma_1 > 0$ with some $\sigma_1 := \sigma_1(\gamma_1,\gamma_2,\alpha) > 0$ and a constant $C' := C'(\gamma_1,\gamma_2,\alpha) > 0$, such that the following Carleman estimate holds true

$$s^3\lambda^4 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i}\varphi_i^3 |q_i|^2 dx\,dt + s\lambda^2 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i}\varphi_i |\partial_x q_i|^2 dx\,dt$$
$$+ s^3\lambda^3 \int_0^T \varphi_1(t,1) e^{-2s\eta_1(t,1)} |q_1(t,1)|^2 dt \le C' \Bigg[\sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \left|\partial_t q_i + \partial_x(\gamma_i \partial_x q_i)\right|^2 dx\,dt$$
$$+ s\lambda \int_0^T \varphi_2(t,0) e^{-2s\eta_2(t,0)} |\partial_x q_2(t,0)|^2 dt \Bigg],$$

for $s \ge s_1$, $\lambda \ge \lambda_1$ and for all $(q_1,q_2) \in \mathcal{Q}$.

Finding the observability inequality

Carleman estimate

$$s^3\lambda^4 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i^3 |q_i|^2 dx\,dt + s\lambda^2 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx\,dt$$
$$+ s^3\lambda^3 \int_0^T \varphi_1(t,1) e^{-2s\eta_1(t,1)} |q_1(t,1)|^2 dt \leq C' \Big[\sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \big|\partial_t q_i + \partial_x(\gamma_i \partial_x q_i)\big|^2 dx\,dt$$
$$+ s\lambda \int_0^T \varphi_2(t,0) e^{-2s\eta_2(t,0)} |\partial_x q_2(t,0)|^2 dt \Big].$$

- The solution q to the adjoint system satisfies

$$\begin{cases} \partial_t q_1 + \partial_x(\gamma_1 \partial_x q_1) = a q_2, \\ \partial_t q_2 + \partial_x(\gamma_2 \partial_x q_2) = 0 \end{cases} \tag{11}$$

- The source integral in the right hand side:

$$\int_0^T \int_0^1 e^{-2s\eta_1} |a q_2|^2 \, dx\,dt \leq 8a^2 T^6 \int_0^T \int_0^1 \varphi_2^3 \, e^{-2s\eta_1} |q_2|^2 \, dx\,dt$$
$$\leq 8a^2 T^6 \int_0^T \int_0^1 \varphi_2^3 \, e^{-2s\eta_2} |q_2|^2 \, dx\,dt =: \widetilde{X}.$$

We used $\beta_2 \geq \beta_1$ and so $\eta_2 \leq \eta_1$ which implies $e^{-2s\eta_2} \geq e^{-2s\eta_1}$ for any $s > 0$.

Observability inequality

Finally, we have

$$
(12)\quad s\lambda^2 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx\, dt + s^3\lambda^3 \int_0^T \varphi_1(t,1) e^{-2s\eta_1(t,1)} |q_1(t,1)|^2 dt \leq Cs\lambda \int_0^T \varphi_2(t,0) e^{-2s\eta_2(t,0)} |\partial_x q_2(t,0)|^2 dt.
$$

From the above inequality, one can obtain the following.

Proposition (Observability inequality)

For any $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}$, the associated solution $q := (q_1, q_2) \in \mathcal{C}^0([0,T];\mathcal{H}) \cap L^2(0,T;(H^2(0,1))^2)$ to the adjoint system satisfies the following observation estimate

$$
(13)\quad \|q(0)\|_{\mathcal{H}}^2 \leq Ce^{C/T} \int_0^T |\partial_x q_2(t,0)|^2 dt,
$$

for some constant $C := C(\gamma_1, \gamma_2, \alpha, a) > 0$ that does not depend on $T > 0$ and ζ.

Outline

Control on y_1: why Carleman trick does NOT work!

- The observation term is $\mathcal{B}_1^* q(t) = \gamma_1(0)\partial_x q_1(t,0)$.
- To obtain a *Carleman estimate*, we need

$$\begin{cases} \beta_i(x) = 2 + c_i(x-1), \quad \forall x \in [0,1], \\ \text{with } c_1 = c_1(\gamma_1,\gamma_2) < 0, \quad c_2 = 1, \\ \beta_1 \geq \beta_2 > 0, \text{ in } [0,1], \ \beta_1(1) = \beta_2(1). \end{cases}$$

- A Carleman estimate still holds with the observation integral

$$s\lambda \int_0^T \varphi_1(t,0) e^{-2s\eta_1(t,0)} |\partial_x q_1(t,0)|^2 dt. \tag{14}$$

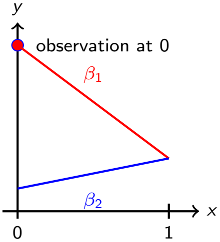


Figure: Typical auxiliary function

- A proper *observability inequality* is lacking! The source term is (in the r.h.s.)

$$\int_0^T \int_0^1 e^{-2s\eta_1} |aq_2|^2 \, dxdt, \tag{15}$$

which cannot be absorbed by

$$s^3\lambda^4 \int_0^T \int_0^1 e^{-2s\eta_2} \varphi_2^3 |q_2|^2 \, dxdt, \quad \text{since } e^{-2s\eta_1} \geq e^{-2s\eta_2} \ (\text{as } \beta_1 \geq \beta_2 \Rightarrow \eta_1 \leq \eta_2). \tag{16}$$

- Indeed, in the next few slides we shall observe that the controllability depends on the parameters in a sensitive way.

Control on y_1: Spectral Analysis

Spectral Analysis: Fix $\gamma_1 = \gamma_2 = 1$, the eigenvalue problem is $\mathcal{A}^*_{\alpha,a} u = \lambda u$ for $\lambda \in \mathbb{C}$,

$$
\begin{cases}
-u_1'' + a u_2 = \lambda u_1 & \text{in } (0,1), \\
-u_2'' = \lambda u_2 & \text{in } (0,1), \\
u_1(0) = 0, \ u_2(0) = 0, \\
u_1(1) = u_2(1), \\
u_1'(1) + u_2'(1) + \alpha u_1(1) = 0.
\end{cases} \tag{17}
$$

- **Localization of the spectrum.** Let any $a \in \mathbb{R}^*$ and $\alpha \geq 0$. Denote the spectrum of $\mathcal{A}^*_{\alpha,a}$ by $\Lambda_{\alpha,a}$. We have

$$\Lambda_{\alpha,a} \subset \bigcup_{\lambda \in \Lambda_\alpha} D(\lambda, 2|a|),$$

where $\Lambda_\alpha := \Lambda_{\alpha,0}$ is the set of all eigenvalues of the self-adjoint operator $\mathcal{A}_\alpha := \mathcal{A}_{\alpha,0}$.

– If $\xi \in \rho(\mathcal{A}_\alpha)$ such that $|\xi - \lambda| \geq 2|a|$ for any $\lambda \in \Lambda_\alpha$, then we see

$$\|(\mathcal{A}_\alpha - \xi I)^{-1}\| = \sup_{\lambda \in \Lambda_\alpha} \frac{1}{|\xi - \lambda|} \leq \frac{1}{2|a|}.$$

It follows that $\xi \in \rho(\mathcal{A}^*_{\alpha,a})$,

$$\mathcal{A}^*_{\alpha,a} - \xi I = \mathcal{A}_\alpha - \xi I + \mathcal{M}^*_a = (\mathcal{A}_\alpha - \xi I)\left(I - (\mathcal{A}_\alpha - \xi I)^{-1}\mathcal{M}^*_a\right),$$

$$\|(\mathcal{A}^*_\alpha - \xi I)^{-1}\mathcal{M}^*_a\| \leq \|(\mathcal{A}^*_\alpha - \xi I)^{-1}\| \|\mathcal{M}^*_a\| \leq \frac{1}{2|a|}|a| < 1.$$

Eigenfunctions of $\mathcal{A}^*_{\alpha,a}$

- **Set of eigenfunctions.**
  - **Case $\lambda = 0$:** This is possible iff $a + 3\alpha + 6 = 0$ and the eigenfunction is
$$\Phi_0(x) = \begin{pmatrix} \frac{ax^3}{6} + (1 - \frac{a}{6})x \\ x \end{pmatrix}, \quad \forall x \in [0,1].$$
  - **Case $\lambda \neq 0$:** The set of eigenfunctions are
$$\Phi_\lambda(x) := \begin{pmatrix} -\frac{ax}{2\mu}\cos(\mu x) + \left(1 + \frac{a}{2\mu}\frac{\cos\mu}{\sin\mu}\right)\sin(\mu x) \\ \sin(\mu x) \end{pmatrix}, \quad \text{with } \mu^2 := \lambda,$$
where $\lambda \in \mathbb{C}$ are the *associated eigenvalues* and $\mu \in \mathbb{C}$ satisfies

(18) $$f(\mu) := (4\mu^2 - a)\sin 2\mu + 2a\mu + 4\alpha u \sin^2 \mu = 0.$$

  - Completeness of $\{\Phi_\lambda\}_{\lambda\in\Lambda_{\alpha,a}}$ in $\mathcal{H}$ (even in L^2) comes as a consequence of a theorem of Keldysh, since the perturbation $\mathcal{M}^*_a$ is bounded.

Description of the spectrum

The set of eigenvalues of $\mathcal{A}_\alpha$ are $\{\lambda^\alpha_{k,1}, \lambda^\alpha_{k,2}\}_{k\geq 0}$ where $\lambda^\alpha_{k,1} \in \big((k+1/2)^2\pi^2, (k+1)^2\pi^2\big)$ and $\lambda^\alpha_{k,2} = (k+1)^2\pi^2$.

Lemma

There exists some $k_{\alpha,a} \in \mathbb{N}$ such that for each $k \geq k_{\alpha,a}$, the operator $\mathcal{A}^*_{\alpha,a}$ has two real eigenvalues $\lambda^{\alpha,a}_{k,1}$ and $\lambda^{\alpha,a}_{k,2}$ in the disk $D(\lambda^\alpha_{k,i}, 2|a|)$, that satisfy

$$\lambda^{\alpha,a}_{k,1} = (k+1/2)^2\pi^2 + (\alpha + a/2) + O_{\alpha,a}(1/k^2), \ \textit{for large } k,$$

$$\lambda^{\alpha,a}_{k,2} = (k+1)^2\pi^2 - a/2 + O_{\alpha,a}(1/k^2), \ \textit{for large } k.$$

Uniqueness of the real eigenvalues in each disk can be proved using a perturbation argument by Kato, 1995.

Conclusion of the structure of $\Lambda_{\alpha,a}$: We deduce that $\Lambda_{\alpha,a} = \Lambda^0_{\alpha,a} \cup \Lambda^\infty_{\alpha,a}$ with $\Lambda^0_{\alpha,a}$ is finite set,

$$\Lambda^0_{\alpha,a} \subset \bigcup_{i=1,2} \ \bigcup_{0\leq k<k_{\alpha,a}} D(\lambda^\alpha_{k,i}, 2|a|),$$

$$\Lambda^\infty_{\alpha,a} := \left\{\lambda^{\alpha,a}_{k,1}, \ k \geq k_{\alpha,a}\right\} \cup \left\{\lambda^{\alpha,a}_{k,2}, \ k \geq k_{\alpha,a}\right\} \subset \mathbb{R}.$$

Structure of the spectrum

We have $\Lambda_{\alpha,a} = \Lambda^0_{\alpha,a} \cup \Lambda^\infty_{\alpha,a}$ with $\Lambda^0_{\alpha,a}$ is finite and $\Lambda^\infty_{\alpha,a}$ is countably infinite,

$$\Lambda^0_{\alpha,a} \subset \bigcup_{i=1,2} \bigcup_{0 \leq k < k_{\alpha,a}} D(\lambda^\alpha_{k,i}, 2|a|),$$

$$\Lambda^\infty_{\alpha,a} := \left\{\lambda^{\alpha,a}_{k,1},\ k \geq k_{\alpha,a}\right\} \cup \left\{\lambda^{\alpha,a}_{k,2},\ k \geq k_{\alpha,a}\right\}.$$

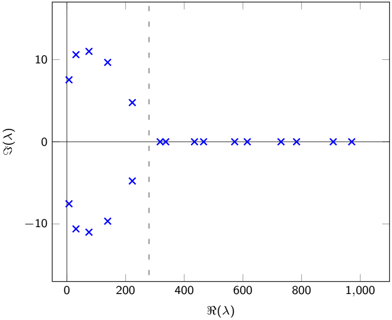


Figure: A numerical description of a part of the spectrum: for $a = 30$, $\alpha = 0.1$

Approximate controllability

We recall the system again:

$$(19)\quad \begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a\, y_1 = 0 & \text{in } (0,T)\times(0,1), \\ y_1(t,0) = v(t), \quad y_2(t,0) = 0 & \text{in } (0,T), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1). \end{cases}$$

Lemma (Approximate controllability)

Let any $a \in \mathbb{R}^*$ and $\alpha \geq 0$ be given. Then there exists a non-empty set $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$, such that we have the following properties:

1. *If $(\alpha, a) \notin \mathcal{R}$, the problem (19) is approximately controllable at any time $T > 0$ in $\mathcal{H}_{-1}$.*
2. *On the other hand, if $(\alpha, a) \in \mathcal{R}$, there exists a subspace $\mathcal{Y}_{\alpha,a} \subset \mathcal{H}_{-1}$ of codimension 1, such that the problem (19) is approximately controllable at any time $T > 0$ if and only if the initial data belongs to $\mathcal{Y}_{\alpha,a}$.*

Approximate controllability (sketch of the proof)

- Let $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ and $\lambda \neq 0$. We compute

(20) $$\mathcal{B}_1^* \Phi_\lambda = -\frac{a}{2\mu} + \mu + \frac{a \cos \mu}{2 \sin \mu}, \quad \mu^2 = \lambda \in \Lambda_{\alpha,a}.$$

Suppose that $\mathcal{B}_1^* \Phi_\lambda = 0$. Since $\mu \neq 0$ and $\sin \mu \neq 0$, this is equivalent to

(21) $$(2\mu^2 - a) \sin \mu + a\mu \cos \mu = 0.$$

The equation satisfied by the eigenvalues,

(22) $$(4\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha u \sin^2 \mu = 0.$$

- The equations (21)–(22) can be represented as

(23) $$\begin{pmatrix} (4\mu^2 - a) & 4\alpha\mu \\ (2\mu^2 - a) & -2a\mu \end{pmatrix} \begin{pmatrix} \sin 2\mu \\ \sin^2 \mu \end{pmatrix} = \begin{pmatrix} -2a\mu \\ -2a\mu \end{pmatrix}.$$

- The determinant of the coefficient matrix: $\det M_\mu = 2a\mu(a + 2\alpha) - 8\mu^3(a + \alpha) \neq 0$,

(24) $$\begin{cases} \sin 2\mu = \dfrac{2a\mu(a + 2\alpha)}{a(a + 2\alpha) - 4\mu^2(a + \alpha)}, \\ \sin^2 \mu = \dfrac{-4a\mu^2}{2a(a + 2\alpha) - 8\mu^2(a + \alpha)}. \end{cases}$$

- Using $\sin^2 2\mu = 4 \sin^2 \mu(1 - \sin^2 \mu)$, we conclude that this situation can only occur for $\mu = \mu^c_{\alpha,a} := \sqrt{a^2 + 2a\alpha + 2a}/2$, i.e., if $\mathcal{B}_1^* \Phi_\lambda = 0$, then we necessarily have

$$\lambda = \lambda^c_{\alpha,a} := a(a + 2\alpha + 2)/4.$$

Now, we need to collect those pairs (α, a) for which $\lambda^c_{\alpha,a}$ does satisfy (24), i.e.,

$$\sin(\sqrt{a^2+2a\alpha+2a}) = \frac{a(a+2\alpha)\sqrt{a^2+2a\alpha+2a}}{a(a+2\alpha)-(a+\alpha)(a^2+2a\alpha+2a)}, \tag{25a}$$

$$\sin^2\left(\frac{\sqrt{a^2+2a\alpha+2a}}{2}\right) = \frac{-a(a^2+2a\alpha+2a)}{2a(a+2\alpha)-2(a+\alpha)(a^2+2a\alpha+2a)}, \tag{25b}$$

Any solution of (25b) necessarily satisfy

$$\sin(\sqrt{a^2+2a\alpha+2a}) = \varepsilon\frac{a(a+2\alpha)\sqrt{a^2+2a\alpha+2a}}{a(a+2\alpha)-(a+\alpha)(a^2+2a\alpha+2a)} \tag{26}$$

with $\varepsilon = \{1, -1\}$.
The blue curves are the solutions of (25b) that satisfy (25a).

This leads to the critical set $\mathcal{R}$ as follows

$\mathcal{R} := \{(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$, such that (25) holds $\}$.

The blue dot is a critical pair $(\alpha, a) \approx (1, 3.1931469)$.

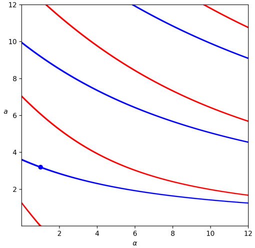


Figure: The set $\mathcal{R}$ of critical pairs (α, a)

Approximate controllability (Fattorini-Hautus criterion). To summarize: for a given pair $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$,

- if $(\alpha, a) \notin \mathcal{R}$, the system is *approximately controllable* in the whole space $\mathcal{H}_{-1}$.
- But as soon as $(\alpha, a) \in \mathcal{R}$, we have $\mathcal{B}_1^* \Phi_{\lambda^c_{\alpha,a}} = 0$ for the critical eigenvalue $\lambda^c_{\alpha,a}$, and in this case the system is *approximately controllable* in

$$\mathcal{Y}_{\alpha,a} := \left\{ y_0 \in \mathcal{H}_{-1} \mid \left\langle y_0, \Phi_{\lambda^c_{\alpha,a}} \right\rangle_{\mathcal{H}_{-1},\mathcal{H}} = 0 \right\} \subset \mathcal{H}_{-1}.$$

Null-controllability: moments approach. The main ingredients are

- We have a proper spectral gap and the required condition on counting function.
- *Observation estimates:* When $(\alpha, a) \notin \mathcal{R}$,

(27) $$|\mathcal{B}_1^* \Phi_\lambda| \geq \frac{1}{C_{\alpha,a}} (1 + \sqrt{|\lambda|}), \quad \forall \lambda \in \Lambda_{\alpha,a}.$$

  For $(\alpha, a) \in \mathcal{R}$, the same estimate is true $\forall \lambda \in \Lambda_{\alpha,a} \setminus \{\lambda^c_{\alpha,a}\}$.
- *Bound on eigenfunctions:*

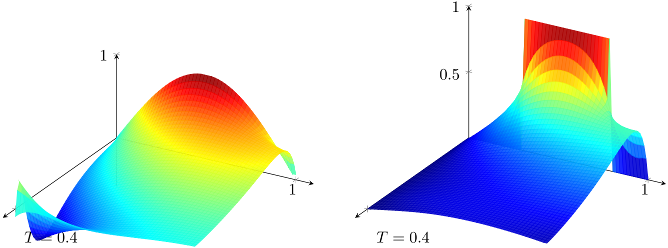
(28) $$\|\Phi_\lambda\|_{\mathcal{H}} \leq C_{\alpha,a} (1 + \sqrt{|\lambda|}), \quad \forall \lambda \in \Lambda_{\alpha,a}$$

- **Conclusion:** Null-controllability on $\mathcal{H}_{-1}$ for $(\alpha, a) \notin \mathcal{R}$ and on $\mathcal{Y}_{\alpha,a}$ for $(\alpha, a) \in \mathcal{R}$.

Evolution of the (HUM) controlled solutions ($(\alpha, a) \in \mathcal{R}$)

$$\begin{cases} T = 0.4, \quad \gamma_1 = \gamma_2 = 1, \quad \alpha_c = 1, a_c = 3.1931469, \\ y_{0,1}(x) = \sin(\pi x), \quad y_{0,2}(x) = \mathbb{1}_{(0.3,0.8)}(x). \end{cases}$$

- Implicit Euler scheme in time • Finite-difference in space
- The penalized HUM approach due to Glowinski, Lions (1995, 2008), Boyer (2012)
- **Control on y_1.**

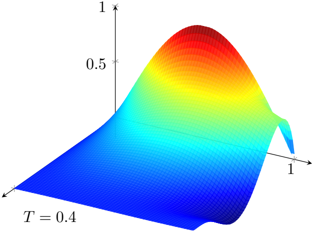


(a) The state $(t, x) \mapsto y_1(t, x)$ (b) The state $(t, x) \mapsto y_2(t, x)$

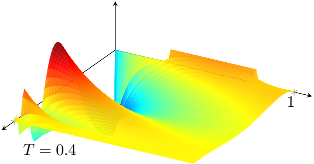
Figure: Evolution in time of the controlled solution: control on y_1.

- **Control on y_2:**

$$\begin{cases} T = 0.4, \quad \gamma_1 = \gamma_2 = 1, \quad \alpha_c = 1, a_c = 3.1931469, \\ y_{0,1}(x) = \sin(\pi x), \quad y_{0,2}(x) = \mathbb{1}_{(0.3,0.8)}(x). \end{cases}$$



(a) The state $(t, x) \mapsto y_1(t, x)$



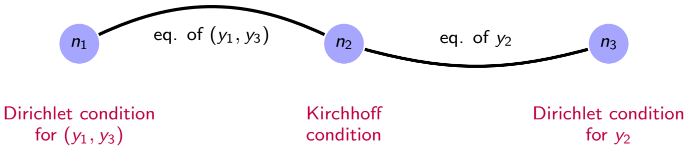
(b) The state $(t, x) \mapsto y_2(t, x)$

Figure: Evolution in time of the controlled solution: control on y_2.

Outline

A 3 × 3 system with one boundary control force

[Bhandari (Thesis), 2020]



Consider the following system for $a \neq 0$ and $\alpha \geq 0$,

$$
(29) \quad \begin{cases}
\partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0, T) \times (0, 1), \\
\partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) = 0 & \text{in } (0, T) \times (0, 1), \\
\partial_t y_3 - \partial_x(\gamma_3 \partial_x y_3) + a\, y_1 = 0 & \text{in } (0, T) \times (0, 1), \\
\begin{cases} y_1(t,1) = y_2(t,1) = y_3(t,1), \\ \displaystyle\sum_{i=1}^{3} \gamma_i(1) \partial_x y_i(t,1) + \alpha y_1(t,1) = 0 \end{cases} & \text{in } (0, T), \\
y_i(0,x) = y_{0,i}(x), \quad \text{for } i = 1,2,3, & \text{in } (0,1),
\end{cases}
$$

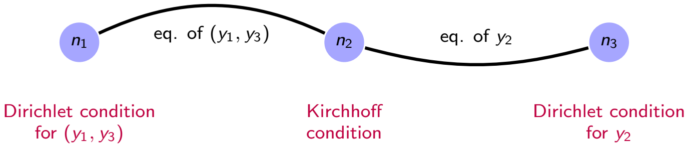
with the following three kind of situations:

$$(30a) \quad y_1(t,0) = v(t), \quad y_2(t,0) = 0, \quad y_3(t,0) = 0 \quad \text{in } (0,T), \quad \text{or}$$

$$(30b) \quad y_1(t,0) = 0, \quad y_2(t,0) = v(t), \quad y_3(t,0) = 0 \quad \text{in } (0,T), \quad \text{or}$$

$$(30c) \quad y_1(t,0) = 0, \quad y_2(t,0) = 0, \quad y_3(t,0) = v(t) \quad \text{in } (0,T).$$

A 3 × 3 system with one boundary control force



Controllability results:

- **Control on the 1st component (Node n_1) – conditionally controllable:** There exists a non-empty set $\widetilde{\mathcal{R}} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$ such that
  1. for each pair $(\alpha, a) \notin \widetilde{\mathcal{R}}$, the system (29)–(30a) is *null-controllable* at any time $T > 0$, for any given data $y_0 \in \widetilde{\mathcal{H}}_{-1} \subset (H^{-1}(0,1))^3$.
  2. for each pair $(\alpha, a) \in \widetilde{\mathcal{R}}$, there exists a subspace $\widetilde{\mathcal{Y}}_{\alpha,a} \subset \widetilde{\mathcal{H}}_{-1}$ of co-dimension 1, such that our system (29)–(30a) is *null-controllable* at any time $T > 0$, for any given data $y_0 \in \widetilde{\mathcal{Y}}_{\alpha,a}$.
- **Control on the 2nd component (Node n_3) – always controllable:** For any $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$, the system (29)–(30b) is *null-controllable* at any time $T > 0$, for any given data $y_0 \in \widetilde{\mathcal{H}}_{-1}$.
- **Control on the 3rd component (Node n_1) – never controllable:** The system (29) with a control on y_3, is NOT even approximately controllable.

Further problem(s)/ result(s)

- What happens if we choose the coupling coefficient $a \in L^\infty(0,1)$ in the 3×3 model!
- **Some result when a control is applied on y_2.** A minimal time for null-controllability is appearing when $\int_0^1 a(x)\,dx = 0$ (BUT $a \neq 0$ a.e.).
- **Control on y_3.** Not even approximately controllable.
- **Control on y_1.** A careful investigation is needed!

Thank You Very Much !