Control of linear parabolic equations

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Webinar Control in Times of Crisis
March, 25, 2021

The aim of this talk is to give an overview on controllability of parabolic equations. Of course, my objective is more restrictive: I will consider only **null controllability of *some* linear and parabolic equations**

What we mean by control of linear parabolic equations

$$\begin{cases} y' = -Ly + Bu \\ y(0) = y^0 \in H. \end{cases}$$

1. H, U Hilbert spaces and $(-L, D(L))$ generator of a $\mathcal{C}_0$ semigroup on H
2. B is an admissible control operator for the semigroup:

$$B \in \mathcal{L}(U, D(L^*)'),\ \mathrm{R}\,(L_T) \subset H, \quad L_T u = \int_0^T e^{-(T-s)L} Bu(s)\, ds$$

3. $\sigma(L^*) = \Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$ satisfying

$$\Re(\lambda_k) \geq \delta\,|\lambda_k| > 0, \quad \forall k \geq 1,\ \Re(\lambda_k) \underset{k \to +\infty}{\to} +\infty$$

   with finite geometric multiplicity, $r_k < +\infty$
4. $\Phi := \{\phi_{k,j}\}_{k \in \mathbb{N}^*,\, 1 \leq j \leq r_k}$, eigenvectors of L^* associated to Λ is a complete family of H

Goal

To define an application

$$(\Lambda, \Phi, B^*) \mapsto T_{inf}$$

where $T_{inf} \in [0, +\infty]$ is defined by

$$\forall T > T_{inf}, (L, B) \text{ is null controllable at time } T,$$

$$\forall T < T_{inf}, (L, B) \text{ is } \textbf{not} \text{ null controllable at time } T.$$

The pair $(L, B)$is null controllable at time $T > 0$ if

$$\forall y^0 \in H, \exists\, u \in L^2((0,T); U) \text{ such that } y(T; y^0, u) = 0$$

Controllability-Observability Inequality

Null-controllability at time T

$$R(e^{-TL}) \subset R(L_T) \qquad \Longleftrightarrow$$

$$\exists\, C_T > 0,\ \forall z \in H, \quad \|e^{-TL^*} z\|^2 \leq C_T \int_0^T \|B^* e^{-tL^*} z\|_U^2\, dt$$

Observability inequality at time T

$$L_T u = \int_0^T e^{-(T-s)L} B u(s)\, ds$$

Resolvent inequality "Quantified" Fattorini-Hautus test

$$\begin{cases} y'(t) = -Ly(t) + Bu(t), & t \geq 0, \\ y(0) = y_0. \end{cases}$$

T. DUYCKAERTS AND L. MILLER, *Resolvent conditions for the control of parabolic equations*, J. Funct. Anal (2012)

Theorem

If the pair (L, B) is null controllable at time T, then, there exists $C_T > 0$ such that for any $y \in D(L^*)$, for any $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, one has,

$$\|y\|^2 \leq C_T e^{2T\Re(\lambda)} \left(\frac{\|(L^* - \lambda)y\|^2}{\Re(\lambda)^2} + \frac{\|B^* y\|_U^2}{\Re(\lambda)} \right).$$

Controllability-Observability Inequality-Resolvent Inequality

Null-controllability at time T

$$R(L_T) \subset R(e^{-TL}) \qquad \Longleftrightarrow$$

$$\exists\, C_T > 0,\ \forall z \in H, \quad \|e^{-TL^*} z\|^2 \leq C_T \int_0^T \|B^* e^{-tL^*} z\|_U^2\, dt$$

Observability inequality at time T

$$\Downarrow$$

$$\exists C_T > 0,\ \forall \lambda \in \mathbb{C},\ \Re\lambda > 0,\ \forall y \in D(L^*),$$

$$\|y\|^2 \leq C_T e^{2T\Re(\lambda)} \left(\frac{\|(L^* - \lambda)y\|^2}{\Re(\lambda)^2} + \frac{\|B^* y\|_U^2}{\Re(\lambda)} \right)$$

Resolvent inequality at time T

A lower bound for T_{inf}

Definition (F. Ammar Khodja, A. B, M. González-Burgos, M. Morancey, 2017)

$$T_0 := \inf\Big\{ T > 0\,;\ \exists C_T > 0; \forall y \in D(L^*), \forall \lambda \in \mathbb{C} \text{ with } \Re(\lambda) > 0,$$

$$\|y\|^2 \leq C_T e^{2T\Re(\lambda)} \left(\frac{\|(L^* - \lambda)y\|^2}{\Re(\lambda)^2} + \frac{\|B^* y\|_U^2}{\Re(\lambda)} \right) \Big\}$$

and $T_0 = +\infty$ when the previous set is empty.

$$T_{inf} \geq T_0$$

Let us analyze T_0 in order to obtain information on T_{inf}

Time of observation

If the system is null controllable at time T, then $T \geq T_0$ with

$$T_0 := \inf\Big\{ T > 0\,;\ \exists C_T > 0; \forall y \in D(L^*), \forall \lambda \in \mathbb{C} \text{ with } \Re(\lambda) > 0,$$
$$\|y\|^2 \leq C_T e^{2T\Re(\lambda)} \left(\frac{\|(L^* - \lambda)y\|^2}{\Re(\lambda)^2} + \frac{\|B^* y\|_U^2}{\Re(\lambda)} \right) \Big\}$$

This implies that for all eigenvalues $\lambda \in \sigma(L^*)$, for all ϕ_λ, normalized eigenvector of L^*, for all $\varepsilon > 0$:

$$\|B^* \phi_\lambda\|_U^2 \geq C_\varepsilon e^{-2(T_0+\varepsilon)\Re(\lambda)}$$

Then

$$T_0 \geq \limsup_{\Re(\lambda) \to +\infty} \frac{-\ln\left(\|B^* \phi_\lambda\|_U\right)}{\Re(\lambda)}$$

In case of geometric multiplicity greater than one

Let r_λ the geometric multiplicity of $\lambda \in \sigma(L^*)$.

$$\inf_{0 \neq y \in E_\lambda} \frac{\|B^* y\|_U^2}{\|y\|^2} \geq C_\varepsilon \Re(\lambda) e^{-2\Re(\lambda)(T_0+\varepsilon)}$$

Let σ_λ the smallest eigenvalue of the Gram matrix $G_\lambda = (\langle B^*\phi_{\lambda,i}, B^*\phi_{\lambda,j}\rangle_U)_{1\leq i\leq j\leq r_\lambda}$.Then,

Time of observation a lower bound for T_{inf}

$$T_{\inf} \geq T_{obs} := \limsup_{\Re(\lambda)\to+\infty} \frac{-\ln\left(\sigma_\lambda\right)}{2\Re(\lambda)}$$

$$\forall \lambda \in \sigma(L^*),\ r_\lambda = 1 \Longrightarrow T_{obs} = \limsup_{\Re(\lambda)\to+\infty} \frac{-\ln\left(\|B^*\phi_\lambda\|\right)}{\Re(\lambda)}$$

Time of observation

- $L = -\partial_{xx}$, $D(L) = H^2(0,\pi) \cap H_0^1(0,\pi)$, $B = 1_{(a,b)}$, $0 < a < b < \pi$.

$$\inf_{k\geq 1} \|B^*\phi_k\|_U^2 = C \inf_{k\geq 1} \int_a^b \sin^2(kx)\,dx > 0 \Longrightarrow T_{obs} = 0$$

- Guided waves: $\Omega = (0,1) \times (0,H)$ (Y. Dermenjian)

$$L = -c\Delta,\ D(L) = \{u \in H_0^1(\Omega), c\Delta u \in L^2(\Omega)\}, c(x) = \begin{cases} 1, & x_2 \in (0,\frac{1}{2}) \\ 2, & x_2 \in (\frac{1}{2},H) \end{cases}$$

$$\frac{1}{2} < a < H, \quad \int_a^H \int_0^1 \phi_k^2(x)\,dx_1\,dx_2 \sim Ce^{-C\left(\lambda_k\right)^{\frac{1}{4}}(a-\frac{1}{2})}$$

$$B = 1_{(a,H)} \Longrightarrow \|B^*\phi_k\| \sim e^{-C\left(\lambda_k\right)^{\frac{1}{4}}} \Longrightarrow \lim_{k\to+\infty} \frac{-\ln\left(\|B^*\phi_k\|\right)}{\lambda_k} = 0$$

$$" \Longrightarrow " T_{obs} = 0$$

Time of observation: the example of Grushin

$$\begin{cases} \partial_t y = \partial_{x_1x_1} y + x_1^2 \partial_{x_2x_2} y + \mathbf{1}_\omega u(t,x_1,x_2), & (t,(x_1,x_2)) \in (0,T) \times \Omega, \\ y(t,x_1,x_2) = 0, & (t,(x_1,x_2)) \in (0,T) \times \partial\Omega, \\ y(0,x_1,x_2) = y_0(x_1,x_2), \end{cases}$$

Theorem

Let $\Omega := (-1,1) \times (0,1)$, $0 < a < b < 1$.

1. *If* $\omega = (-b,a) \times (0,1)$, *the system is null controllable for all* $T > 0$
2. *If* $\omega = (a,b) \times (0,1)$, *the system is null controllable at time* T *if and only if* $T > \frac{a^2}{2}$.

Note that in this case,

$$T_{inf} = \frac{a^2}{2}$$

Time of observation: the example of Grusshin

K. BEAUCHARD, P. CANNARSA, AND R. GUGLIELMI, *Null controllability of Grushin-type operators in dimension two*, J. Eur. Math. Soc. (2014)

K. BEAUCHARD, L. MILLER, AND M. MORANCEY, *2D Grushin-type equations: minimal time and null controllable data*, J. Differential Equations, (2015)

Time of observation: the example of internal control of Grushin

$\omega = (a, b) \times (0, 1)$

There exists a sequence $(\lambda_n, \phi_n)_{n\in\mathbb{N}^*}$ of eigenvalues, eigenvectors, of $L^* = -\partial_{x_1x_1} - x_1^2\partial_{x_2x_2}$ such that

- $$\lambda_n \sim n\pi$$

- The eigenvectors concentrate outside ω:

$$\|B^*\phi_n\|^2 = \int_\omega \phi_n^2(x_1, x_2)dx_1\, dx_2 \underset{n\to+\infty}{\sim} \frac{e^{-a^2\lambda_n}}{2a\pi\sqrt{n}}$$

Thus

$$T_{obs} \geq \frac{a^2}{2} > 0 \Longrightarrow T_{inf} \geq \frac{a^2}{2}$$

Time of observation: the example of boundary control Grushin

K. BEAUCHARD, J. DARDÉ AND S. EREVEDOZA, *Minimal time issues for the observability of Grushin-type equations* , Ann. Inst. Fourier, (2020)

$$\Omega = (-\alpha, \beta) \times (0, \pi), L^* = -\partial_{x_1 x_1} - q^2(x_1)\partial_{x_2 x_2}, B^* z = \partial_{x_1} z(\beta, x_2)$$

$$q(0) = 0, \ , q \in \mathcal{C}^3([-\alpha, \beta]), \inf_{x_2 \in [-\alpha, \beta]} q'(x_1) > 0.$$

- There exist $c > 0$ and a sequence $(\lambda_n, \phi_n)_{n \in \mathbb{N}^*}$ of eigenvalues, eigenvectors, of L^* such that

$$\lambda_n \sim nq'(0), \quad \int_0^\pi |\partial_{x_1} \phi_n(\beta, x_2)|^2 \, dx_2 \le Ce^{-2n\left(\int_0^\beta q(s)\, ds - \varepsilon \right)}$$

Thus

$$T_{obs} \ge \frac{\int_0^\beta q(s)\, ds}{q'(0)} > 0 \Longrightarrow T_{inf} \ge \frac{\int_0^\beta q(s)\, ds}{q'(0)}$$

Time of observation: the pointwise control of the heat equation

S. Dolecki , *Observability for the one-dimensional heat equation*, Studia Math. (1973)

- $$T_{obs} = \limsup_{k \to +\infty} \frac{-\ln(|B^* \phi_k|)}{k^2} = \limsup_{k \to +\infty} \frac{-\ln(|\sin(kx_0)|)}{k^2} = T(x_0)$$

- $$\forall \tau \in [0, +\infty], \ \exists x_0 \in (0, \pi), \ \ T(x_0) = \tau$$

$$\begin{cases} \partial_t y - \partial^2_{xx} y = u(t)\delta_{x_0} & \text{in } (0,\pi) \times (0,T) \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T) \\ y(\cdot,0) = y^0 & \text{in } (0,\pi) \end{cases}$$

In all these previous cases it has been proved that $T_{obs} = T_{inf}$

Recall:

$$\forall T > T_{inf}, (L, B) \text{ is null controllable at time } T,$$

$$\forall T < T_{inf}, (L, B) \text{ is } \textbf{not} \text{ null controllable at time } T.$$

Let σ_λ the smallest eigenvalue of the Gram matrix

$$G_\lambda = \Big(\langle B^* \phi_{\lambda,i}, B^* \phi_{\lambda,j} \rangle_U \Big)_{1 \leq i \leq j \leq r_\lambda}$$

Then,

Time of observation a lower bound for T_{inf}

$$T_{\text{inf}} \geq T_{obs} := \limsup_{\Re(\lambda) \to +\infty} \frac{-\ln(\sigma_\lambda)}{2\Re(\lambda)}$$

Note that this time comes from considering the restriction of the resolvent inequality to $\bigcup_{\lambda \in \sigma(L^*)} E_\lambda$

$E_\lambda \oplus E_\mu$

$$\|y\|^2 \le C_T e^{2T\Re(\lambda)} \left(\frac{\|(L^* - \lambda)y\|^2}{\Re(\lambda)^2} + \frac{\|B^* y\|_U^2}{\Re(\lambda)} \right)$$

Consider the restriction of the resolvent inequality to $\bigcup_{\lambda \neq \mu \in \sigma(L^*)} E_\lambda \oplus E_\mu$

$$T_0 \ge \sup_{\lambda \neq \mu \in \sigma(L^*), 0 \neq y \in E_\lambda \oplus E_\mu} \frac{-\ln\left(\frac{|\lambda-\mu|^2 \|P_{E_\mu} y\|^2}{\Re(\lambda)\|y\|^2} + \frac{\|B^* y\|_U^2}{\|y\|^2} \right)}{2\Re(\lambda)}$$

Two quantities

This lower bound of T_0 depends on these two quantities where $y \in E_\lambda \oplus E_\mu$

1. $$\frac{|\lambda - \mu|^2 \|P_{E_\mu} y\|^2}{\Re(\lambda)\|y\|^2}$$

2. $$\frac{\|B^* y\|_U^2}{\|y\|^2}$$

For this lower bound to be strictly positive, even infinite, it is necessary that the two quantities go to 0.

Let us begin by considering the case where $B^* y = 0$. This is possible when U, the space of control, is of dimension 1 and more generally, this is possible if

$$\ker B^* \cap E_\lambda \oplus E_\mu \neq \{0\}$$

$\ker B^* \cap E_\lambda \oplus E_\mu \neq \{0\}$

For simplicity , let us assume that the eigenvalues are simple.
There exists $b_{\lambda,\mu} \in \mathbb{R}$ such that $B^*\left(\phi_\lambda + b_{\lambda,\mu}\phi_\mu\right) = 0$. Then

$$T_0 \geq \sup_{\lambda \neq \mu \in \sigma(L^*),\, \phi_\lambda + b_{\lambda,\mu}\phi_\mu \in \ker B^* \setminus \{0\}} \frac{-\ln\left(\frac{|\lambda-\mu|^2 b_{\lambda,\mu}^2}{\Re(\lambda)\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\|^2}\right)}{2\Re(\lambda)}$$

T_0 can be strictly positive, even infinite if

- There exists a sequence (λ_k, μ_k) of distinct eigenvalues of L^* such that $\liminf |\lambda_k - \mu_k| = 0$. This is an **eigenvalues-condensation phenomena**
- There exists a sequence (λ_k, μ_k) of distinct eigenvalues of L^* such that $\liminf b_{\lambda_k,\mu_k} = 0$. This comes from the time of observation.

$$B^*\left(\phi_\lambda + b_{\lambda,\mu}\phi_\mu\right) = 0 \Longrightarrow \|B^*\phi_\lambda\| = |b_{\lambda,\mu}|\|B^*\phi_\mu\|$$

$\ker B^* \cap E_\lambda \oplus E_\mu \neq \{0\}$

Let us focus on the condensation. So, we suppose that

$$\sup_{\lambda \neq \mu \in \sigma(L^*),\, \phi_\lambda + b_{\lambda,\mu}\phi_\mu \in \ker B^* \setminus \{0\}} \frac{-\ln\left(\frac{b_{\lambda,\mu}}{\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\|}\right)}{\Re(\lambda)} = 0$$

Then

$$T_0 \geq T_{condensation} := \sup_{\lambda \neq \mu \in \sigma(L^*),\, \phi_\lambda + b_{\lambda,\mu}\phi_\mu \in \ker B^* \setminus \{0\}} \frac{-\ln\left(\frac{|\lambda - \mu|}{\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\|}\right)}{\Re(\lambda)}$$

Condensation of eigenelements

$$T_{condensation} := \sup_{\lambda \neq \mu \in \sigma(L^*), \phi_\lambda + b_{\lambda,\mu}\phi_\mu \in \ker B^* \setminus \{0\}} \frac{-\ln\left(\frac{|\lambda-\mu|}{\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\|}\right)}{\Re(\lambda)}$$

The *condensation time*, $T_{condensation}$ depends on the behavior of the following two quantities

1. $|\lambda - \mu|$
2. $\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\|$

More precisely, $T_{condensation}$ depends on the condensation of the eigenvalues and the condensation of the eigenvectors.

Condensation of eigenelements

Theorem

If $\Phi = \{\phi_\lambda\}_{\lambda\in\sigma(L^*)}$ *is a Riesz basis, then there exists* $c > 0$ *such that for all* $\lambda, \mu \in \sigma(L^*)$

$$\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\| \geq c > 0.$$

Riesz basis avoid eigenvectors-condensation phenomena

In this case, the *condensation time* depends only on the condensation of the eigenvalues:

$$T_{condensation} = \sup_{\lambda\neq\mu\in\sigma(L^*)} \frac{-\ln\Big(|\lambda-\mu|\Big)}{\Re(\lambda)}$$

Condensation of eigenvalues: boundary control of coupled parabolic equations

F. AMMAR KHODJA, A.B, M. GONZÁLEZ-BURGOS, L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal (2014)

$$\begin{cases} \partial_t y = \left(D\partial_{xx}^2 - A\right) y & \text{in } Q_T \\ y(0,\cdot) = Bu, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B^* z = \mathcal{B}^* \partial_x z(0)$$

$$\Lambda = \{\lambda_{k,1} = k^2,\ \lambda_{k,2} = dk^2,\ k \geq 1\}, \phi_{k,i}(x) = V_{k,i} \sin(kx),\ , i = 1, 2,\ , k \geq 1$$

Condensation of eigenvalues: boundary control of coupled parabolic equations

- $\sqrt{d} \notin \mathbb{Q}$
- $\Phi = \{\phi_{k,i}\}_{k\geq 1,\, i=1,2}$ is a Riesz Basis
- $|B^*\phi_{k,i}| \sim Ck \Rightarrow T_{obs} = 0$
- $$B^*\left(\frac{\phi_{k,1}}{B^*\phi_{k,1}} - \frac{\phi_{j,2}}{B^*\phi_{j,2}}\right) = 0$$

Let j_k the nearest integer to $\frac{k}{\sqrt{d}}$, thus:

$$T_{inf} \geq T_0 \geq T_{condensation} = \limsup_{k\to+\infty} \frac{-\ln|k^2 - dj_k^2|}{k^2}$$

$$\forall \tau \in [0, +\infty],\ \exists \sqrt{d} \in \mathbb{R}\setminus\mathbb{Q} :$$

$$\limsup_{k\to+\infty} \frac{-\ln|k^2 - dj_k^2|}{k^2} = \tau = T_{inf}$$

Condensation of eigenelements

What happens if the eigenfunctions do not form a Riesz basis ?

$$T_{condensation} = \sup_{\lambda \neq \mu \in \sigma(L^*),\, \phi_\lambda + b_{\lambda,\mu}\phi_\mu \in \ker B^* \setminus \{0\}} \frac{-\ln\left(\frac{|\lambda-\mu|}{\|\phi_\lambda + b_{\lambda,\mu}\phi_\mu\|}\right)}{\Re(\lambda)}$$

Let me show you the example which was at the origin of the interest in this question.

Condensation of eigenvalues an eigenfunctions: boundary control of coupled parabolic equations

E.H SAMB, *Boundary null-controllability of two coupled parabolic equations: simultaneous condensation of eigenvalues and eigenfunctions*, ESAIM: COCV, (2021)

$$\begin{cases} \partial_t y = \left(D\partial_{xx}^2 - qA\right) y & \text{in } Q_T \\ y(0,\cdot) = Bu, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B^* z = \mathcal{B}^* \partial_x z(0)$$

$$q \in L^\infty(0,\pi), \quad \Lambda = \{\lambda_{k,1} = k^2,\ \lambda_{k,2} = dk^2,\ k \geq 1\}$$

Condensation of eigenvalues: boundary control of coupled parabolic equations

- There exists $d < 1$ such that $\{\phi_{k,i}\}_{k\geq 1,\, i=1,2}$ is not a Riesz Basis

Thus, in *the condensation time*, one cannot avoid a condensation eigenvectors phenomena:

$$T_0 \geq T_{condensation} = \limsup_{k\to+\infty} \frac{\ln \frac{\left\| \frac{\phi_{k,1}}{B^*\phi_{k,1}} - \frac{\phi_{j_k,2}}{B^*\phi_{j_k,2}} \right\|}{|k^2 - dj_k^2|}}{k^2}$$

where j_k is the nearest integer to $\frac{k}{\sqrt{d}}$

In his paper, E.H. Samb has proved that

$$T_{inf} = \max\{T_{obs}, T_{condensation}\}$$

Academic example: condensation of eigenvalues without condensation of eigenvectors

F. AMMAR KHODJA, A. B, C. DUPAIX ET I. KOSTIN, *Null controllability of some systems of parabolic type by one control force*, ESAIM: COCV, (2005)

Let $(\mu_k) \subset]0,+\infty[$, $\mu_{k+1}-\mu_k \geq c > 0$, $\mu_k \underset{k\to+\infty}{\to} +\infty$,

$$A = \sum_k \mu_k(.,\xi_k)\xi_k, \qquad 0 < f(s) \underset{s\to+\infty}{=} o(s)$$

$$L = \begin{pmatrix} A & -f(A) \\ -f(A) & A \end{pmatrix}, \quad D(L) = D(A) \times D(A)$$

$$B : u \in U = X \mapsto (0,u) \in X \times X = H$$

$$\Lambda = \{\lambda_{k,1} := \mu_k - f(\mu_k), \lambda_{k,2} := \mu_k + f(\mu_k),\ k \geq 1\}$$

$$\phi_{k,1} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xi_k, \quad \phi_{k,2} := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xi_k.$$

Academic example

- $T_{obs} = 0$
- $B^*\left(\phi_{k,1} - \phi_{k,2}\right) = 0$
- Φ is Hilbert basis
- $\lambda_{k,1} - \lambda_{k,2} = 2f(\mu_k) \underset{k\to+\infty}{\to} 0$

$$T_{inf} \geq T_0 \geq T_{condensation} = \limsup_{k\to+\infty} \frac{-\ln(f(\mu_k))}{\mu_k}.$$

Note that it has been proved that

$$T_{inf} = T_{condensation}$$

Academic example

A. B, F. BOYER AND M. MORANCEY, *A block moments method to handle spectral condensation phenomenon in parabolic control problems*, Annales Henri Lebesgue, Vol. 3, pp. 717-793 (2020)

Let $(\mu_k) \subset]0, +\infty[$, $\mu_{k+1} - \mu_k \geq c > 0$, $\mu_k \underset{k \to +\infty}{\to} +\infty$,

$$A = \sum_k \mu_k (., \xi_k)\xi_k, \qquad 0 < f(s) \underset{s \to +\infty}{=} o(s)$$

$$L = \begin{pmatrix} A & -I \\ 0 & A - f(A) \end{pmatrix}, \; D(L) = D(A) \times D(A), \quad B : u \in H \mapsto (0, u) \in H \times H$$

$$\Lambda = \{\lambda_{k,1} := \mu_k, \lambda_{k,2} := \mu_k + f(\mu_k), \; k \geq 1\}$$

$$\phi_{k,1} := \frac{1}{\sqrt{1 + f(\mu_k)^2}} \begin{pmatrix} -f(\mu_k) \\ 1 \end{pmatrix} \xi_k, \quad \phi_{k,2} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_k,$$

Academic example

- $T_{obs} = 0$
- $B^*\left(\phi_{k,1} - \phi_{k,2}\right) = 0$
- Condensation of eigenvalues:

$$|\lambda_{k,1} - \lambda_{k,2}| = |f(\mu_k)| \underset{k \to \infty}{\longrightarrow} 0$$

- Condensation of eigenfunctions:

$$\|\phi_{k,2} - \phi_{k,1}\|^2 = \frac{2\left(1 + f(\mu_k)^2 - \sqrt{1 + f(\mu_k)^2}\right)}{1 + f(\mu_k)^2} \sim f(\mu_k)^2 \underset{k \to \infty}{\longrightarrow} 0$$

The condensation of the eigenfunctions compensate the condensation of the eigenvalues. Then

$$T_{condensation} = 0$$

$$\sum_{k \geq 1} \frac{1}{\mu_k} < +\infty \Longrightarrow T_{inf} = T_{condensation} = 0$$

Return to the resolvent inequality

$$T_0 \geq \sup_{\lambda \neq \mu \in \sigma(L^*),\, 0 \neq y \in E_\lambda \oplus E_\mu} \frac{-\ln\left(\frac{|\lambda-\mu|^2 \|P_{E_\mu} y\|^2}{\Re(\lambda)\|y\|^2} + \frac{\|B^* y\|_U^2}{\|y\|^2}\right)}{2\Re(\lambda)}$$

What happens if

$$\ker B^* \cap E_\lambda \oplus E_\mu = \{0\}?$$

Suppose that Φ is a Hilbert basis and let $\sigma_{\lambda,\mu}$ the smallest eigenvalue of the Gram matrix associated to the family $\{B^*\phi_{\lambda,i}, B^*\phi_{\mu,j}, 1 \leq i \leq r_\lambda,\ 1 \leq j \leq r_\mu\}$. Then,

$$T_0 \geq \sup_{\lambda \neq \mu \in \sigma(L^*)} \frac{-\ln\left(|\lambda - \mu| + \sqrt{\sigma_{\lambda,\mu}}\right)}{\Re(\lambda)}$$

As previously, I propose to analyze this situation through an example.

Condensation and observation of the eigenfunctions

L.OUAILI, *Minimal time of null controllability of two parabolic equations*, Math. Control Relat.Fields 10 (2020), no. 1, 89-112.

$$\omega = (a, b) \subset (0, 1)$$

$$\begin{cases} \partial_t y = -Ly + 1_\omega \mathcal{B}\, u & \text{in } Q_T := (0,T) \times (0,1), \\ y(\cdot, 1) = y(\cdot, 0) = 0 & \text{on } (0,T), \\ y(0, \cdot) = y_0 & \text{in } (0,1), \end{cases}$$

$$L = -\begin{pmatrix} \partial_{xx} + \mu_1\, q & 0 \\ 0 & \partial_{xx} + \mu_2\, q \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$q \in L^2(0,1),\ \mu_1 \neq \mu_2 \in \mathbb{R},\ \ b_1 b_2 \neq 0$$

$$B = 1_{(a,b)} \mathcal{B}$$

Condensation and observation of the eigenfunctions

$$L_i\psi_{k,i} = \Big(- \partial_{xx} - \mu_i q\Big)\psi_{k,i} = \lambda_{k,i}\psi_{k,i}, \;\; \sigma(L^*) = \Lambda = \{\lambda_{k,1}, \lambda_{k,2},\, k \geq 1\}$$

$$\phi_{k,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_{k,1}, \;\; \phi_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \psi_{k,2}$$

Supp $q \cap \omega = \emptyset$

Assume Supp $q \subset (0, a)$ or Supp $q \subset (b, 1)$. Then,

$$\sigma_k \sim |\lambda_{k,1} - \lambda_{k,2}|^2,$$

where σ_k is the smallest eigenvalue of the Gram matrix
$G_k = (\langle B^*\phi_{k,i}, B^*\phi_{k,j}\rangle_U)_{1\leq i\leq j\leq 2} = (\langle \phi_{k,i}, \phi_{k,j}\rangle_{L^2(\omega)})_{1\leq i\leq j\leq 2}$.

$$T_{inf} \geq \limsup_{k\to+\infty} \frac{-\ln|\lambda_{k,1} - \lambda_{k,2}|}{\lambda_{k,1}}$$

Condensation and observation of the eigenfunctions

Theorem (L. Ouaili)

Assume that

$$\operatorname{Supp} q \subset (0,a) \text{ or } \operatorname{Supp} q \subset (b,1),$$

Then,

$$T_{inf} = \limsup_{k\to+\infty} \frac{-\ln|\lambda_{k,1}-\lambda_{k,2}|}{\lambda_{k,1}}.$$

M. González-Burgos & L. de Teresa, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force* Port. Math, (2010)

If Supp $q \cap \omega \neq \emptyset$...

The system is null controllable at any time, even if

$$\limsup_{k\to+\infty} \frac{-\ln|\lambda_{k,1}-\lambda_{k,2}|}{\lambda_{k,1}} > 0$$

Resolvent inequality $\cap$ $E_\lambda \oplus E_\mu$

$E_\lambda \oplus E_\mu$

$$T_0 \geq \sup_{\lambda \neq \mu \in \sigma(L^*), 0 \neq y \in E_\lambda \oplus E_\mu} \frac{-\ln\left(\frac{|\lambda-\mu|^2}{\Re(\lambda)\|y\|^2} + \frac{\|B^* y\|_U^2}{\|y\|^2}\right)}{2\Re(\lambda)}$$

It takes into account the *condensation* of TWO eigenelements.

Resolvent inequality $\cap \bigoplus_{1\leq i\leq N} E_{\lambda_i}$

$$\|y\|^2 \leq C_T e^{2T\Re(\lambda)} \left(\frac{\|(L^* - \lambda)y\|^2}{\Re(\lambda)^2} + \frac{\|B^* y\|_U^2}{\Re(\lambda)} \right)$$

The restriction of the resolvent inequality to $\bigoplus_{1\leq i\leq N} E_{\lambda_i}$ with $N > 2$ is not relevant. It takes into account

$$\max_{1\leq i\leq N-1} (\lambda_N - \lambda_i)$$

while the condensation of the eigenvalues may be bigger as we will see.

The condensation index

V.L. BERNSTEIN *Leçons sur les progrès récents de la théorie des séries de Dirichlet* (1933)
J.R. SHACKELL *Overconvergence of Dirichlet series with complex exponents* J. Analyse Math., 22:135-170, (1969)

Assume that Λ satisfies

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

The interpolating function is defined by

$$E_\Lambda : z \in \mathbb{C} \mapsto \prod_{\lambda \in \Lambda} \left(1 - \frac{z^2}{\lambda^2}\right).$$

The condensation index $c(\Lambda)$ is defined by

$$c(\Lambda) = \limsup_{\substack{\lambda \in \Lambda \\ \lambda \to \infty}} \frac{-\ln |E'_\Lambda(\lambda)|}{\lambda}.$$

The Condensation index

F. Ammar Khodja, A. B, M. González-Burgos, L. de Teresa, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal (2014)

Theorem

Assume... and assume

$$T_{obs} = \limsup_{k \to +\infty} \frac{\ln \frac{1}{|B^* \phi_k|}}{\Re(\lambda_k)} = 0,$$

then

$$T_{inf} = c(\Lambda)$$

The condensation index

Let $\alpha > \beta > 0$ and $\Lambda = \{k^2, k^2 + e^{-\alpha k^2}, k^2 + e^{-\beta k^2}\ ,\ k \in \mathbb{N}^*\}$. We can prove that

$$c(\Lambda) = \alpha + \beta$$

Then,

$$T_{inf} = \alpha + \beta$$

But

$$\lim_{k \to +\infty} \frac{-\ln\left(\max\{|\lambda_{1,k} - \lambda_{2,k}|,\ |\lambda_{1,k} - \lambda_{3,k}|\}\right)}{k^2} = \alpha < \alpha + \beta$$

The lower bound given by the resolvent inequality is not optimale

Conclusion?

- The resolvent inequality does not cover all the situations.
- Nevertheless it allowed us to highlight some, all (?) *ingredients* which enter into account in the definition of T_{inf}.
- Franck Boyer has conjectured a modified resolvent inequality that at least can take into account every finite combination of eigenspaces...work in progress. The precision of constants is fundamental. As you have seen, the presence of $e^{\Re(\lambda)T}$ in the resolvent inequality is crucial.

Thanks to Morgan Morancey because he is the first to understand the interest of this coefficient!

Return to the goal

Goal

To define an application

$$(\Lambda, \Phi, B^*) \mapsto T_{inf}$$

Is it possible to define a such application?

I do not know. But, if we add some assumptions to the ones at the start, the answer is :YES

A. B, F. Boyer and M. Morancey, *A block moments method to handle spectral condensation phenomenon in parabolic control problems*, Annales Henri Lebesgue, (2020)

Additonnal assumptions in the case where $\Lambda \subset \mathbb{R}$

- $\dim U = 1$
- Asymptotic behavior:

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty$$

- Weak gap condition: $\exists (\rho, p) \in]0, +\infty[\times \mathbb{N}$,

$$\#\Big(\Lambda \cap [\mu, \mu + \rho]\Big) \leq p, \quad \forall \mu \in [0, +\infty)$$

The class $\mathcal{G}(\Lambda, p, r, \rho)$

Let $p \in \mathbb{N}$, $r, \rho \in]0, +\infty[$ given.
$(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda, p, r, \rho)$, if

- $$\Lambda = \bigcup_{k\geq 1} G_k, \qquad G_k \cap G_j, \ \forall k \neq j,$$
- For every $k \geq 1$,
  $$n_k := \# G_k \leq p,$$
- $$\mathrm{dist}(G_k, G_{k+1}) \geq r,$$
- $$\mathrm{diam}\, G_k < \rho$$

Return to the goal of this talk

If, $\Lambda \subset \mathbb{R}$, $\dim U = 1$, $\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty$ & weak-gap, then

$$(\Lambda, \Phi, B^*) \mapsto T_{inf} := \limsup_{k \to \infty} \frac{\ln \left(\max\limits_{l \in \{1,\dots,n_k\}} \| \sum\limits_{1 \le j \le l} \frac{\xi_{k,j}}{\prod\limits_{1 \le i \neq j \le l} (\lambda_{k,i} - \lambda_{k,j})} \| \right)}{\lambda_{k,1}}$$

$$\Lambda = \bigcup_{k \ge 1} G_k, \ \ G_k = \{\lambda_{k,1}, \cdots, \lambda_{k,n_k}\}$$

$$\xi_{k,j} := \frac{\phi_{k,j}}{B^* \phi_{k,j}}$$

Comments

Comments

- What about non scalar controls? F. Boyer & M. Morancey in a near future
- What about the space dimension greater than one?
  - One of the interests of the resolvent inequality is that it does not require the convergence of the series $\sum \frac{1}{\lambda}$. It is therefore no restricted to the dimension one. Finding the right resolvent inequality should allow us to define the expression of T_{inf}
  - Knowing the cost of the control makes it possible to obtain results in dimensions greater than 1 in the case of tensorized operators. The following recent result is a big step in this direction:

M. GONZÁLEZ-BURGOS & L. OUAILI, *Sharp estimates for biorthogonal families to exponential functions associated to complex sequences without gap conditions*, (2020), https://hal.archives-ouvertes.fr/hal-03115544/

Thank you for your attention