Remarks on the Cauchy problem for the Laplacian and the Approximate Lagrangian controllability of the Euler Equation

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Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a regular boundary $\partial\Omega$, and let Γ be a part of $\partial\Omega$ with nonempty relative interior. Assume that a subdomain $\omega \subset\subset \Omega$ is given such that its boundary $\gamma := \partial\omega$ is a Jordan curve and let us denote by $\mathbf{n}$ the exterior normal to the boundary of $\Omega \setminus \overline{\omega}$. The question we address in this talk is the following: given a *target* function h defined on γ, can one find a *control* function v defined on $\partial\Omega$ having its support $\mathrm{supp}(v) \subset \Gamma$, and such that the solution Ψ of

$$\Delta\Psi = 0 \quad \text{in } \Omega, \qquad \frac{\partial\Psi}{\partial\mathbf{n}} = v \quad \text{on } \partial\Omega, \tag{1}$$

satsifies

$$\frac{\partial\Psi}{\partial\mathbf{n}} = h \quad \text{on } \gamma \qquad \text{or} \qquad \left\| \frac{\partial\Psi}{\partial\mathbf{n}} - h \right\|_{H^{-1/2}(\gamma)} \leq \varepsilon \ ? \tag{2}$$

(Here $H^{-1/2}(\gamma)$ denotes the Sobolev space of order $-1/2$ on γ). The motivation of this question lies in its application to the approximate Lagrangian control of Euler equation. The same question is studied when $\Omega \subset \mathbb{R}^N$ is a sooth bounded domain and $N \geq 2$.

We will also address the following related ill-posed problem: let $0 < R_1 < R_0$ two positive numbers and $\Omega := \{x \in \mathbb{R}^2 \ ; \ R_1 < |x| < R_0\}$. Let

$$\Gamma_j := [|x| = R_j] \quad \text{for} \quad j = 0, 1,$$

and for $(f_0, g_0) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ we consider the following Cauchy problem:

$$-\Delta u = 0 \quad \text{in} \ \ \Omega,$$

with

$$u = 0 \quad \text{on} \ \ \Gamma_1, \qquad u = f_0 \quad \text{on} \ \ \Gamma_0 \qquad \frac{\partial u}{\partial \mathbf{n}} = g_0 \quad \text{on} \ \ \Gamma_0.$$

We describe a Hilbert space $\mathbb{H} \subset H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ such that the above problem has a unique solution $u \in H_0^1(\Omega)$ if, and only if, $(f_0, g_0) \in \mathbb{H}$ and for two constants $c_1, c_2 > 0$ we have

$$c_1 \|(f_0, g_0)\|_{\mathbb{H}} \leq \|u\|_{H_0^1(\Omega)} \leq c_2 \|(f_0, g_0)\|_{\mathbb{H}}.$$

An analogous result holds when Ω is the rectangular domain $\Omega := (0, \pi) \times (0, \ell) \subset \mathbb{R}^2$ for some $\ell > 0$, while $\Gamma_0 := [0, \pi] \times \{0\}$ and $\Gamma_1 := \{0\} \times [0, \ell] \cup \{\pi\} \times [0, \ell]$. We give necessary and sufficient conditions on the Cauchy data f_0, g_0 so that there exists $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω and

$$u = 0 \quad \text{on} \ \ \Gamma_1, \qquad u = f_0 \quad \text{on} \ \ \Gamma_0 \quad \text{and} \ \ \frac{\partial u}{\partial \mathbf{n}} = g_0 \quad \text{on} \ \ \Gamma_0. \tag{3}$$