Boundary controllability of a one-dimensional phase-field system with one control force

We use the moment method to present a controllability result for nonlinear phase-field systems of Caginalp type when the scalar control force v acts on the temperature equation of the system.

$$\begin{cases} \tilde{\theta}_t - \xi\tilde{\theta}_{xx} + \dfrac{1}{2}\rho\xi\tilde{\phi}_{xx} + \dfrac{\rho}{\tau}\tilde{\theta} = -\dfrac{\rho}{4\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right) & \text{in } Q_T := (0,\pi)\times(0,T), \\ \tilde{\phi}_t - \xi\tilde{\phi}_{xx} - \dfrac{2}{\tau}\tilde{\theta} = \dfrac{1}{2\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right) & \text{in } Q_T, \\ \tilde{\theta}(0,\cdot) = v,\ \tilde{\phi}(0,\cdot) = c,\ \tilde{\theta}(\pi,\cdot) = 0,\ \tilde{\phi}(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde{\theta}(\cdot,0) = \tilde{\theta}_0,\ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{in } (0,\pi). \end{cases}$$

Our objective is to prove the null controllability result at time T for the temperature $\tilde{\theta}$ but keeping the material in solid state ($c = 1$), or liquid state, ($c = -1$) at time T, that is to say, proving that there exists a control $v \in L^2(0,T)$ such that system above has a solution $\tilde{y} = (\tilde{\theta}, \tilde{\phi})$ (in an appropriate space) such that

$$\tilde{\theta}(\cdot,T) = 0 \quad \text{and} \quad \tilde{\phi}(\cdot,T) = c \quad \text{in } (0,\pi).$$